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# Kinks Inside Supersymmetric Domain Ribbons

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## ABSTRACT

We study a variety of supersymmetric systems describing sixth-order interactions between two coupled real superfields in  $2+1$  dimensions. We search for BPS domain ribbon solutions describing minimum energy static field configurations that break one half of the supersymmetries. We then use the supersymmetric system to investigate the behavior of mesons and fermions in the background of the defects. In particular, we show that certain BPS domain ribbons admit internal structure in the form of bosonic kinks and fermionic condensate, for a given range of the two parameters that completely identify the class of systems.

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# 1 Introduction

The idea of topological defects that present internal structure was first introduced in [1], within the context of modeling superconducting strings. It was also explored by other authors [2, 3] in different contexts, and more recently the works [4, 5] also investigate systems that admit the existence of defects inside topological defects. General features of the works just mentioned are that they consider systems in  $3 + 1$  dimensions, and that the potential describing the scalar fields depends on several parameters. In this case, solutions representing topological defects with internal structure only appear after adjusting some of the several parameters that defines the model under consideration.

In some recent works [6, 7, 8], solitons that emerge in certain bidimensional systems of coupled real scalar fields were studied. The class of systems considered in those works has been shown [9, 10] to admit a natural embedding into the bosonic sector of a supersymmetric theory, in such a way that the set of free parameters is quite restricted. Within this context, the existence of topological defects inside domain walls in a model of two real scalar field belonging to the above mentioned class of systems was first considered in [9]. There, the system is defined by a potential that contains up to the quartic power in the scalar fields, and the set of parameters is reduced to just two parameters. The system has a domain wall solution that traps in its interior a topological defect produced by the remaining scalar field, provided the parameters ratio is positive definite. It also has solutions known as domain ribbons inside a domain wall, and they resemble a stringlike configuration that can be either infinitely long or in the form of a closed loop.

As pointed out in [9], the reduction in the number of free parameters may perhaps lead to a clearer understanding of the physical properties these kind of systems can comprise. This is one of the main motivation of the present paper, in which we shall further explore the possible existence of defects inside topological defects in systems belonging to the class of systems already introduced in [6, 8]. Here, however, we shall investigate systems containing up to the sixth power in the scalar fields. In this case we shall restrict our investigation to  $2+1$  dimensional spacetime, and so we are going to search for kinks inside supersymmetric domain ribbons. There are many

reasons to consider such systems, among them we would like to single out the following: Potentials with sixth-order interactions define systems that admit the existence of solitons of different nature, at least in the sense that they may connect adjacent vacua in a richer set of vacuum states. As we are going to show below, there are at least three systems that seems to be worth investigating, one of them was already considered in [6], in the 1+1 dimensional case, and the others will be defined below.

The investigations are organized as follows. In the next Section we perform the construction of the class of systems of our interest in the framework of supersymmetric field theory. In Sec. 3, we show that these systems present a Bogomol'nyi bound for the energy whose saturation is achieved provided the fields solve a set of first-order equations, simpler than the usual equations of motion. The solutions of these Bogomol'nyi equations break one half of the supersymmetries and then belong to a short supermultiplet. In Sec. 4 we illustrate the procedure by investigating some specific systems. Sec. 4.1 is devoted to the study of a model already introduced in [6], and there we show that this system does not allow the formation of kinks neither the trapping of mesons inside the domain ribbon. This result can be traced up to an asymmetry produced by the fact that the domain ribbon found for the sixth order system connects the symmetric vacuum state with non-symmetric ground states. Nevertheless, the effective potential owed to this configuration favor the entrapping of Majorana fermions inside the domain ribbon. In Sec. 4.2 we present a simple extension of the previous model that circumvent the above mentioned asymmetry allowing the existence of a kink inside the supersymmetric domain ribbon. We briefly discuss on the formation of these kinks and their stability. The third model is presented in Sec. 4.3, and its main feature is that both superfields present equal footing from the beginning. Then, we show that the system admits domain ribbons with an internal kink of the same sixth-order nature, for a given range of parameters, and we briefly discuss its classical stability. Also, we investigate the behaviour of fermions in the background of these solutions in Sec. 4.4. We end this paper in Sec. 5, where we introduce some comments, further remarks and conclusions.

## 2 Supersymmetric Systems of Coupled Real Scalars

A general system of two real scalar fields in  $2 + 1$  spacetime is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - U(\phi, \chi), \quad (1)$$

where  $U(\phi, \chi)$  is the potential, in general a nonlinear function of the two fields  $\phi$  and  $\chi$  involving several coupling constants for the different terms. As explained above, an interesting framework that highly restricts the dimensionality of the parameter space is provided by supersymmetry. In that respect, let us start by considering a supersymmetric field theory in  $(2 + 1)$  spacetime dimensions, entirely constructed from two real superfields [11]  $\Phi$  and  $\Xi$ ,

$$\Phi = (\phi, \psi, D_\phi) \quad \Xi = (\chi, \rho, D_\chi), \quad (2)$$

where  $\psi$  and  $\rho$  are Majorana two-spinors, while  $D_\phi$  and  $D_\chi$  are bosonic auxiliary fields. The Lagrangian density can be written in terms of the superfields as

$$\mathcal{L}_{N=1} = \frac{1}{2} \int d^2\theta \left[ \bar{\mathcal{D}}\Phi \mathcal{D}\Phi + \bar{\mathcal{D}}\Xi \mathcal{D}\Xi + \mathcal{W}(\Phi, \Xi) \right]. \quad (3)$$

Here we are following the conventions introduced in [12] and, as usual,  $\mathcal{D}$  is the supercovariant derivative

$$\mathcal{D} = \partial_{\bar{\theta}} + i\bar{\theta}^\mu \partial_\mu, \quad (4)$$

with the  $\gamma$ -matrices being represented by  $\gamma^0 = \tau^3$ ,  $\gamma^1 = i\tau^1$  and  $\gamma^2 = -i\tau^2$ . In terms of component fields, after replacing the auxiliary fields  $D_\phi$  and  $D_\chi$  by their algebraic equations of motion, the Lagrangian density can be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{i}{2} \bar{\psi} \not{\partial} \psi + \frac{i}{2} \bar{\rho} \not{\partial} \rho - \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \phi} \right)^2 \\ &- \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \chi} \right)^2 - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi^2} \bar{\psi} \psi - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial \chi^2} \bar{\rho} \rho - \frac{\partial^2 \mathcal{W}}{\partial \phi \partial \chi} \bar{\psi} \rho, \end{aligned} \quad (5)$$

where it is explicit that the Yukawa couplings as well as the scalar potential of the theory entirely depends on the superpotential  $\mathcal{W}$ . In fact, we stress that the scalar potential results to be

$$U(\phi, \chi) = \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \phi} \right)^2 + \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \chi} \right)^2, \quad (6)$$

in direct correspondence with a general class of systems that comprises quite interesting properties as was already described in Refs.[6, 7, 8, 9]. The set of transformations that leave invariant the system described by (5) is

$$\delta_\eta \phi = \bar{\eta} \psi \quad \delta_\eta \chi = \bar{\eta} \rho, \quad (7)$$

$$\delta_\eta \psi = \left( -i \not{\partial} \phi + \frac{\partial \mathcal{W}}{\partial \phi} \right) \eta, \quad (8)$$

$$\delta_\eta \rho = \left( -i \not{\partial} \chi + \frac{\partial \mathcal{W}}{\partial \chi} \right) \eta, \quad (9)$$

where the infinitesimal parameter  $\eta$  is a real spinor. Let us now focus upon the classical configurations of this system. We will then set the fermion fields to zero and look for purely bosonic field configurations which, from the point of view of the supersymmetric theory, can be understood as background solutions. Responses of the fermion fields to these backgrounds can then be investigated. We then introduce the following useful notation: Given a functional  $\mathcal{F}$  depending both on bosonic and fermionic fields, we will use  $\mathcal{F}|$  to refer to that functional evaluated in the purely bosonic background,

$$\mathcal{F}| \equiv \mathcal{F}|_{\rho, \psi=0}. \quad (10)$$

Under condition (10), the only non-vanishing supersymmetric transformations that leave invariant the Lagrangian (5) are those corresponding to fermionic fields.

### 3 BPS Domain Ribbons

Let us now show that the supersymmetric nature of the system imposes lower bounds for the mass per unit length of a generic bosonic static configuration

that is homogenous in one of the spatial coordinates. Indeed, one can compute the conserved supercharge that generates the transformations (8) and (9) to be

$$\mathcal{Q}_\alpha = \frac{1}{2} \int d^2x \left\{ \left[ -i \not{\partial} \phi + \frac{\partial \mathcal{W}}{\partial \phi} \right] \psi_\alpha + \left[ -i \not{\partial} \chi + \frac{\partial \mathcal{W}}{\partial \chi} \right] \rho_\alpha \right\} , \quad (11)$$

and use it to construct the supercharge algebra over the static bosonic background resulting

$$\{ \mathcal{Q}_\alpha, \mathcal{Q}_\beta \} = \gamma_{\alpha\beta}^0 M + \epsilon_{\alpha\beta} Z , \quad (12)$$

where  $M$  is the mass of the purely bosonic configuration

$$M = \frac{1}{2} \int d^2x \left[ (\vec{\nabla} \phi)^2 + (\vec{\nabla} \chi)^2 + \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \phi} \right)^2 + \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \chi} \right)^2 \right] , \quad (13)$$

whereas the ‘central extension’ of the algebra  $Z$  is given by a line integral over a curve  $\Gamma$  that encloses the region where the fields carry a finite energy density

$$Z = \oint_\Gamma \vec{\nabla} \mathcal{W} \cdot d\vec{x} . \quad (14)$$

It is clear from its expression that the ‘central extension’  $Z$  is in general forced to vanish as a consequence of the scalar nature of the real superpotential. Indeed, the supersymmetry algebra of any three-dimensional system with an  $N = 1$  invariance does not have room for the introduction of a central charge.

There is a breakthrough which consists in studying configurations that are independent of one of the spatial coordinates, say  $x_2$ . Then, in order to deal with finite quantities that make sense, one must reinterpret  $\mathcal{Q}$  as the supercharge per unit length and also  $M$  as a uniform longitudinal mass density. In this case, the closed curve  $\Gamma$  should be identified with a discrete set of two points asymptotically located at both infinities in the  $x_1$ -axis. Then, the equation (14) must be rewritten as

$$Z = \mathcal{W}(x_1 \rightarrow \infty) - \mathcal{W}(x_1 \rightarrow -\infty) \equiv \Delta \mathcal{W} , \quad (15)$$

and the supercharge algebra (12) should be understood as the  $N = 2$ ,  $d = 2$  supersymmetry algebra which, in fact, admits the appearance of central

extensions. Now, the positive-definiteness of the supercharge algebra (12) implies

$$M \geq |\Delta\mathcal{W}| . \quad (16)$$

This is nothing but the Bogomol'nyi bound of the coupled real scalar system introduced above. Indeed, although  $\Delta\mathcal{W}$  is apparently different from the usual definition of the topological charge  $T$ , it actually coincides with it since both depend only on the topology [13]. It can be seen that  $\Delta\mathcal{W}$  vanishes in a topologically trivial state and has a positive value in a domain ribbon state. The appearance of the topological charge as a central extension of the supercharge algebra seems to contradict previous results obtained in Ref.[14] for kink states in  $d = 2$  systems. However, we must point out that the configurations we are considering fit into a dimensional reduction scheme, and so the result (15) can be seen as an expected result [15, 16].

It is straightforward to see that the configurations that saturate the bound (16) are those preserving one half of the supersymmetries. The explicit result appears after choosing  $\gamma^0\eta_{\pm} = \pm\eta_{\pm}$ , which allows writing

$$\{ \mathcal{Q}[\eta_{\pm}], \mathcal{Q}[\eta_{\pm}] \} = \frac{1}{2} \int d^2x \left[ (\delta_{\eta_{\pm}}\psi)^{\dagger}(\delta_{\eta_{\pm}}\psi) + (\delta_{\eta_{\pm}}\rho)^{\dagger}(\delta_{\eta_{\pm}}\rho) \right] , \quad (17)$$

and therefore the bound is saturated provided  $\delta_{\eta_{+}}(\psi, \rho) = 0$  or  $\delta_{\eta_{-}}(\psi, \rho) = 0$ , thus preserving the purely bosonic nature of the background configuration. Here we recall that the only configuration that preserves all the supersymmetries is the trivial vacuum configuration. Furthermore, the equations that saturate the lower bound are nothing but the Bogomol'nyi equations of the system:

$$\frac{d\phi}{dx} \pm \frac{\partial\mathcal{W}}{\partial\phi} = 0 , \quad (18)$$

$$\frac{d\chi}{dx} \pm \frac{\partial\mathcal{W}}{\partial\chi} = 0 . \quad (19)$$

Let us consider the case when the supersymmetry corresponding to the parameter  $\eta_{+}$  is unbroken (thus, eqs.(18) and (19) are valid with the upper sign). The supersymmetry generated by  $\eta_{-}$  is broken in the BPS domain ribbon background given by the solution of these equations. The variations (8) and (9) for the broken supersymmetries, as expected, give zero energy Grassmann variations of the domain ribbon solution; that is, they are zero

modes of the Dirac equation in the background of the defect as can be easily verified. Quantization of these fermionic Nambu-Goldstone zero modes leads to the construction of a (BPS) supermultiplet of degenerate bosonic and fermionic soliton states which transform according to a short representation of the supersymmetry algebra.

We have so far given a supersymmetry derived proof of the existence of a Bogomol'nyi bound and self-duality equations in the family of relativistic systems of coupled real scalar fields first introduced in Ref. [6]. In the next Section we will introduce specific systems, which comprise very interesting topological defects that may or may not present internal structure provided the superpotential  $\mathcal{W}$  is conveniently chosen.

## 4 Some Specific Systems

Let us now consider explicit examples of systems of two real scalar fields containing up to the sixth power in the scalar fields. To illustrate the procedure, in the following we will consider three different systems, the first two containing different powers in each one of the fields. This is interesting because we will find defects of different nature corresponding to each one of the two fields. In the third system, both fields  $\phi$  and  $\chi$  enter the game with equal footing, that is, the potential in this case contains sixth order powers in both fields.

### 4.1 BPS $\phi^6$ ribbons without internal kinks

As a first system to study in order to obtain a deeper insight on the topological defects that result from the saturation of the Bogomol'nyi bound, we will consider a superpotential of the form

$$\mathcal{W}_1(\phi, \chi) = \frac{1}{2}\lambda\phi^2\left(\frac{1}{2}\phi^2 - a^2\right) + \frac{1}{2}\mu\phi^2\chi^2. \quad (20)$$

The potential that results from it can be obtained after (6) to be

$$\begin{aligned} U_1(\phi, \chi) = & \frac{1}{2}\lambda^2\phi^2(\phi^2 - a^2)^2 + \lambda\mu\phi^2(\phi^2 - a^2)\chi^2 + \\ & \frac{1}{2}\mu^2\phi^2\chi^4 + \frac{1}{2}\mu^2\phi^4\chi^2. \end{aligned} \quad (21)$$



This potential has an explicit discrete symmetry  $Z_2 \times Z_2$ , and degenerate vacuum states ( $\phi^2 = a^2$ ,  $\chi = 0$ ) that breaks the symmetry  $Z_2$  corresponding to the  $\phi$  field. It also has a flat direction for the vacuum expectation value of  $\chi$  when the scalar field  $\phi$  seats at the symmetric vacuum  $\phi = 0$ . In particular, the minimum ( $\phi = 0$ ,  $\chi = 0$ ) preserves the whole  $Z_2 \times Z_2$  symmetry.

The set of first order differential (Bogomol'nyi) equations corresponding to bosonic configurations of this system is given by

$$\frac{d\phi}{dx} = \lambda\phi(\phi^2 - a^2) + \mu\phi\chi^2, \quad (22)$$

and

$$\frac{d\chi}{dx} = \mu\phi^2\chi. \quad (23)$$

The bosonic sector of this system was already investigated in [6]. There it was found the following set of solutions: The first pair of solutions is  $\chi = 0$ , and

$$\phi^2(x) = \frac{1}{2}a^2[1 - \tanh(\lambda a^2 x)]. \quad (24)$$

It is easy to see that (24) is a solution just by noting that, if one sets  $\chi \rightarrow 0$  in the potential, one gets

$$U_1(\phi, 0) = \frac{1}{2}\lambda^2\phi^2(\phi^2 - a^2)^2, \quad (25)$$

a sixth-order potential which is known to admit solutions of the form (24). The BPS domain ribbon is located at  $x = 0$  and its thickness is given by  $\delta \approx (\lambda a^2)^{-1}$ . It is convenient to regard the domain ribbon as a slab of false vacuum of width  $\delta$  with  $\phi^2 = a^2/2$  in the interior and  $\phi^2 = 0$ ,  $\phi^2 = a^2$  at both sides of it.

A second pair of BPS solutions for the system above is

$$\phi^2(x) = \frac{1}{2}a^2[1 - \tanh(\mu a^2 x)] \quad \text{and} \quad \chi^2(x) = \left(\frac{\lambda}{\mu} - 1\right)\phi^2(x), \quad (26)$$

and in this last case one must require that  $\lambda/\mu > 1$ . Both solutions have the same energy per unit length, which is given by

$$E_1 = \frac{1}{4}|\lambda|a^4, \quad (27)$$

and this follows from the fact that both pairs belong to the same topological sector. Furthermore, both pairs of solutions are classically or linearly stable, and this follows from the general result [7] that ensures stability of solutions that solve the pair of first-order differential equations, that is, of BPS solutions.

For the kink described by (24), let us first point out that this defect connects two regions which are very different from the beginning: a symmetric region with a vanishing value of  $\phi$  is connected to an asymptotic region where the discrete symmetry  $Z_2$  related to the transformation  $\phi \rightarrow -\phi$  is broken. In this sense, this kind of solitons are asymmetric, in contrast to the  $\phi^4$  system, where the kink connects asymptotic regions both having asymmetric vacua [9]. The issue here is that the  $\phi^4$  model is usually considered to simulate second-order phase transitions, where the kink defects describe order-order interfaces that appear in this case. However, the  $\phi^6$  system is related to first-order transitions in which the two phases, symmetric and asymmetric or disordered and ordered, may appear simultaneously. Evidently, in this last case the kink defects describe order-disorder interfaces. As we are going to show, interesting physical consequences may appear in this richer situation where the potential can present up to the sixth power in the fields. To see how this comes out, let us introduce masses for the  $\phi$  mesons living outside the kink:  $m_\phi^2(0,0) = \lambda^2 a^4$  and  $m_\phi^2(a^2,0) = 4\lambda^2 a^4$ , depending upon the side. Furthermore, for  $\phi = 0$  the  $\chi$  mesons appear to be massless, and for  $\phi^2 = a^2$  we obtain

$$U_1(\pm a, \chi) = \frac{1}{2}\mu^2 a^4 \chi^2 + \frac{1}{2}\mu^2 a^2 \chi^4, \quad (28)$$

and the  $\chi$  mesons are such that  $m_\chi^2(a^2,0) = \mu^2 a^4$ . As a consequence, this system presents the interesting feature of containing a topological defect separating the outside regions into two distinct regions, one containing massless  $\chi$  mesons and massive  $\phi$  mesons with mass  $\lambda^2 a^4$ , and the other with massive  $\chi$  and  $\phi$  mesons, with masses  $\mu^2 a^4$  and  $4\lambda^2 a^4$ , respectively. Because of this asymmetry, that makes  $\chi$  mesons to be massless at the symmetric vacuum  $\phi = 0$ , there is no way of making the  $\phi$  field to trap the  $\chi$  field in its interior, to give rise to a topological defect inside a topological defect in two space dimensions. Since the  $\chi$  field is massless for  $\phi = 0$ , there is no other energetic argument left to favor the  $\chi$  field to be inside the ribbon. For instance, the region inside the ribbon is defined with  $x = 0$  and here we get  $\phi^2 = a^2/2$ ,

which changes the above potential to the form

$$U_1(\pm a/\sqrt{2}, \chi) = \frac{1}{4}\mu^2 a^2 \left[ \chi^2 - \frac{1}{2} \left( \frac{\lambda}{\mu} - \frac{1}{2} \right) a^2 \right]^2 + \frac{1}{16}\mu^2 a^6 \left( \frac{\lambda}{\mu} - \frac{1}{4} \right) , \quad (29)$$

which presents spontaneous symmetry breaking for  $\lambda/\mu > 1/2$ . However, spontaneous symmetry breaking requires the presence of massive  $\chi$  mesons inside the ribbon, and these massive mesons would instead decay into the massless mesons that live outside the ribbon. We remark that the case  $\lambda/\mu = 1/2$  seems to be interesting since this would also make the  $\chi$  mesons to be massless inside the domain ribbon, but here spontaneous symmetry breaking would unfortunately not be present anymore.

Before ending this subsection, let us comment a little on stability by following the standard way. From the above potential  $U_1(\pm a/\sqrt{2}, \chi)$  we can write the corresponding kink solutions, for  $\lambda/\mu > 1/2$ ,

$$\chi = \pm \sqrt{\frac{1}{2} \left( \frac{\lambda}{\mu} - \frac{1}{2} \right)} a \tanh \left[ \frac{1}{2} \mu \sqrt{\frac{\lambda}{\mu} - \frac{1}{2}} a^2 y \right] . \quad (30)$$

These solutions, or better the pairs of solutions given by  $\phi^2 = a^2/2$  and  $\chi$  as above, do not solve the first-order equations and so do not give any BPS solution. For this reason the proof of classical stability already introduced in [7, 8] does not work for them. To investigate stability we should consider

$$\phi(y, t) = \bar{\phi} + \sum_n \eta_n(y) \cos(w_n t) , \quad (31)$$

and

$$\chi(y, t) = \bar{\chi} + \sum_n \zeta_n(y) \cos(w_n t) , \quad (32)$$

where  $\bar{\phi}$  and  $\bar{\chi}$  constitute the pair of classical solutions and  $\eta(y)$  and  $\zeta(y)$  are small fluctuations. In this case we substitute the above configurations into the equations of motion to get to the equation

$$S \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} = w_n^2 \begin{pmatrix} \eta_n \\ \zeta_n \end{pmatrix} , \quad (33)$$

valid for small fluctuations and for static classical field configurations. Here  $S$  is the Schrödinger-like operator

$$S = -\frac{d^2}{dy^2} + V(y) , \quad (34)$$

where  $V(y)$  is a  $2 \times 2$  matrix with elements  $V_{ij} = \partial^2 V / \partial f_i \partial f_j$ , where  $f_1 = \phi$  and  $f_2 = \chi$ . Evidently,  $V(y)$  is to be calculated at the static classical configurations  $\bar{\phi}$  and  $\bar{\chi}$  where the small fluctuations are being considered.

For the BPS pair of solutions we have  $\chi = 0$ ; this decouples the fluctuations and allows introducing an explicit analytical investigation, as already done in [6]. For the above non-BPS pair of solutions, however, neither  $\bar{\phi}$  nor  $\bar{\chi}$  vanishes, and so  $V(y)$  does not become diagonal anymore. To get an idea here let us just write the non-diagonal elements of  $V(y)$  in this case:

$$V_{12} = V_{21} = \mu^2 a^4 f(\lambda/\mu) \tanh[g(y)] \{1 + f^2(\lambda/\mu) \tanh^2[g(y)]\} , \quad (35)$$

with  $f(\lambda/\mu) = \sqrt{\lambda/\mu - 1/2}$  and  $g(y) = (1/2)\mu f(\lambda/\mu) a^2 y$ . Experience on former investigations [17] says that the resulting Schrödinger equation does not even map the exactly solvable modified Posch-Teller problem [18], and so a standard investigation concerning classical stability can only be implemented numerically, but this is out of the scope of the present work. Such an investigation should confirm instability of that pair of non-BPS solutions, as suggested by the energy considerations presented above.

## 4.2 $\chi^4$ kinks inside a BPS $\phi^6$ ribbon

Let us now consider another  $\phi^6$  system. We will show in this subsection that it is possible to circumvent the asymmetry outside the domain ribbon of the  $\phi$  field that we have just found in the above system, in spite of the fact that the solution itself is asymmetric. Moreover, we will find that the resulting domain ribbon admits a non-trivial internal structure that can be shown to be a kink. In order to see this, we consider a system described by a modified superpotential  $\mathcal{W}_2$  given by

$$\mathcal{W}_2(\phi, \chi) = \frac{1}{2} \lambda \phi^2 \left( \frac{1}{2} \phi^2 - a^2 \right) + \frac{1}{2} \mu \left( \phi^2 - \frac{1}{2} a^2 \right) \chi^2 . \quad (36)$$

This superpotential gives the following scalar potential

$$U_2(\phi, \chi) = \frac{1}{2}\lambda^2\phi^2(\phi^2 - a^2)^2 + \lambda\mu\phi^2(\phi^2 - a^2)\chi^2 + \frac{1}{2}\mu^2\phi^2\chi^4 + \frac{1}{2}\mu^2\left(\phi^2 - \frac{1}{2}a^2\right)^2\chi^2, \quad (37)$$

that present a quite different vacuum structure without flat directions. The degenerate vacua are given by the fully symmetric state ( $\phi^2 = 0, \chi^2 = 0$ ), a couple of  $Z_2$  invariant states ( $\phi^2 = a^2, \chi^2 = 0$ ) and a set of states ( $\phi^2 = a^2/2, \chi^2 = (\lambda/2\mu)a^2$ ) that break the whole  $Z_2 \times Z_2$  symmetry.

For minimum energy, static field configurations obey the set of first-order equations

$$\frac{d\phi}{dx} = \lambda\phi(\phi^2 - a^2) + \mu\phi\chi^2, \quad (38)$$

and

$$\frac{d\chi}{dx} = \mu\left(\phi^2 - \frac{1}{2}a^2\right)\chi, \quad (39)$$

which present soliton solutions. Indeed, if we set  $\chi = 0$ , it is immediate that this system admits the BPS  $\phi^6$  solution that we found earlier (24), that is,

$$\phi^2(x) = \frac{1}{2}a^2[1 - \tanh(\lambda a^2 x)]. \quad (40)$$

Another pair of solutions can be found using the trial orbit method [19] to be

$$\phi^2(x) = \frac{1}{4}a^2[1 - \tanh(\lambda a^2 x)] \quad \text{and} \quad \chi^2(x) = \frac{1}{2}\phi^2(x), \quad (41)$$

provided the coupling constants obey the relation  $\mu = 2\lambda$ . The above two pair of solutions are BPS solutions and, thus, are stable [7]. As in the former asymmetric case, the  $\phi^6$  domain ribbon connect regions with different transformation properties under the symmetry subgroup  $Z_2$ . However, if one computes the mass of the  $\chi$  mesons at both sides of the domain ribbon (40), it is interesting to see that

$$m_\chi^2(0, \chi) = m_\chi^2(a^2, \chi) = \frac{1}{4}\mu^2 a^4, \quad (42)$$

i.e., the  $\chi$  field does not distinguish between the two outside regions of the BPS  $\phi^6$  ribbon. This new scenario allows building topological defects inside

a topological defect. Indeed, it can be seen that inside the domain ribbon we have  $x = 0$  and this makes  $\phi^2 = a^2/2$ . Consequently, the  $\chi$  mesons feel an effective potential given by

$$U_2(a^2/2, \chi) = \frac{1}{4}\mu^2 a^2 \left( \chi^2 - \frac{\lambda}{2\mu} a^2 \right)^2 . \quad (43)$$

For  $\lambda/\mu > 0$  there is a spontaneous symmetry breaking potential for the  $\chi$  field with vacua states located at  $\chi^2 = \lambda a^2/2\mu$ . Moreover, the mass of the excitations is

$$m_\chi^2(a^2/2, \chi) = \lambda \mu a^4 , \quad (44)$$

and one sees that  $m_\chi^2(out) > m_\chi^2(in)$  for  $\lambda/\mu < 1/4$ , as follows from the above results. This restriction on the parameters ensures that the  $\phi^6$  ribbon traps the  $\chi$  kink in  $2+1$  dimensions. Indeed, it is energetically favorable for the  $\chi$  bosons to remain inside the ribbon: the mass of the boson field  $m(x)$  increases away from the center of the ribbon, resulting in a force  $F \approx -\partial m(x)/\partial x$  that attracts  $\chi$  bosons toward the ribbon, entrapping them. Here we remark that the  $\phi$  defect is of the  $\phi^6$  type, while the  $\chi$  defect is of the  $\chi^4$  type, and so they are of different nature.

In the present case there exists a parameter range  $\lambda/\mu \in (0, 1/4)$ , in which it becomes energetically favorable for a  $\chi$  condensate to form within the core of the domain ribbon. Indeed, the potential (43) is minimized by a field configuration for which  $\chi = \pm\chi_0$  where

$$\chi_0 = \left( \frac{\lambda}{2\mu} \right)^{1/2} a . \quad (45)$$

We take for simplicity  $a$  real and positive. It is straightforward to see that  $U_2(a^2/2, \chi_0) < U_2(a^2/2, 0)$ , and this shows that inside the domain ribbon associated to the  $\phi$  field it is possible that domains where  $\chi = \pm\chi_0$  appear. It is clear that domains with  $\chi_0$  and  $-\chi_0$  should necessarily be connected by topological defects. The interior of the BPS domain ribbon is then a region where the discrete symmetry  $Z_2$  associated with the  $\chi$  field is broken. Thus, inside the domain ribbon scalar condensates will eventually form, but they will be uncorrelated beyond some coherence length  $\xi$  [5]. We therefore expect domains of  $\chi = +\chi_0$  and  $\chi = -\chi_0$  to form at different positions along the

domain ribbon, with each domain extending an average length given by  $\xi$ . Different domains must be separated by a region where  $\chi = 0$  that should be understood as the location of the resulting  $\chi$  kink. This is an example of a topological kink inside the topological domain ribbon. The explicit form of the solution is given by (40) for the host domain ribbon, whereas for the kink we just have

$$\chi(y) = \sqrt{\frac{\lambda}{2\mu}} a \tanh\left(\frac{1}{2}\sqrt{\lambda\mu} a^2 y\right). \quad (46)$$

Evidently, the domain ribbon appears from the pair of BPS solutions given by  $\chi = 0$  and  $\phi$  as in Eq. (40), and is a ribbon, a defect of dimension one in the planar system that we are considering. For the kink, however, we see that it appears *inside* the domain ribbon, which is located at  $x = 0$  and extends along the direction described by  $y$  in the present case. The kink is a topological defect of dimension zero and appears after setting  $\phi^2 = a^2/2$  and removing the  $x$  degree of freedom in the above system.

Investigations concerning classical stability may be introduced by just following the steps already presented in the former subsection. Like there, no analytical result can be obtained in the present case too. Here, however, the modification introduced in the potential allows the appearance of a region in parameter space obeying  $0 < \lambda/\mu < 1/4$ , in which the above pair of solutions might be classically or linearly stable. In this region, as one knows two consecutive kinks must be separated by an antikink, so that the initial kink-antikink separation distance should be of the order of the coherence length  $\xi$ . To determine the explicit form of this coherence length we see that the width of the above kink solution is given by  $1/\sqrt{\lambda\mu}a^2$ , which essentially measures the distance between consecutive domains of  $\chi^2 = \chi_0^2$ , and so should be identified with the coherence length in the present case, that is,  $\xi \approx 1/\sqrt{\lambda\mu}a^2$ .

### 4.3 $\chi^6$ kinks inside a BPS $\phi^6$ ribbon

Let us now consider a third system of coupled real scalar fields that, in spite of displaying an asymmetry of the effective potential outside the domain ribbon, allows the formation of a non-trivial internal structure inside the defect. As we shall see, this system presents a structure that gives a kink of

the  $\chi^6$  type, that is, of the same nature of the host domain ribbon. To see how this works explicitly, let us consider the following superpotential

$$\mathcal{W}_3(\phi, \chi) = \frac{1}{2}\lambda\phi^2\left(\frac{1}{2}\phi^2 - a^2\right) - \frac{1}{2}\mu\chi^2\left(\frac{1}{8}\chi^2 - a^2\right) + \frac{1}{2}\mu\phi^2\chi^2. \quad (47)$$

Here we see that both fields are very similar, although the symmetry is still  $Z_2 \times Z_2$ . The scalar potential generated by this superpotential is given by

$$\begin{aligned} U_3(\phi, \chi) = & \frac{1}{2}\lambda^2\phi^2(\phi^2 - a^2)^2 + \frac{1}{2}\mu^2\chi^2\left(\frac{\chi^2}{4} - a^2\right)^2 + \frac{1}{4}\mu^2\phi^2\chi^4 + \\ & \frac{1}{2}\mu^2\left(1 + \frac{2\lambda}{\mu}\right)\phi^4\chi^2 + \mu^2\left(1 - \frac{\lambda}{\mu}\right)a^2\phi^2\chi^2. \end{aligned} \quad (48)$$

It is clear from the above expression that this system displays spontaneous symmetry breaking for each one of the two field, separately. To see this explicitly, we note that

$$U_3(\phi, 0) = \frac{1}{2}\lambda^2\phi^2(\phi^2 - a^2)^2, \quad (49)$$

and

$$U_3(0, \chi) = \frac{1}{2}\mu^2\chi^2\left(\frac{1}{4}\chi^2 - a^2\right)^2. \quad (50)$$

Thus, a domain ribbon solution of the type previously studied exists for each one of the non-vanishing fields. In order to see which field the system chooses to host the kink, we have to investigate the energy of kinks generated by the corresponding 1 + 1 dimensional systems described by the above potentials. For a more complete investigation concerning this point see Ref. [20], where the high temperature effects on the class of systems of interest are presented<sup>1</sup>. For the system defined by  $U_3(\phi, 0)$  the energy of the corresponding  $\phi^6$  kink is

$$E_{\phi\text{-kink}} = \frac{1}{4}|\lambda|a^4. \quad (51)$$

For the system defined by  $U_3(0, \chi)$  we get

$$E_{\chi\text{-kink}} = |\mu|a^4, \quad (52)$$

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<sup>1</sup>These thermal effects are relevant to the standard cosmological scenario for the formation of the host domain ribbons, since one knows that the cosmic evolution occurs via expansion and cooling.



and so we can write the result

$$E_{\phi\text{-kink}} = \frac{|\lambda|}{4|\mu|} E_{\chi\text{-kink}}. \quad (53)$$

Thus, if we choose to work with  $|\lambda| \geq 4|\mu|$  we can consider the  $\phi$  field as the field to generate the host domain ribbon.

In general, for static solutions the first order equations are given by

$$\frac{d\phi}{dx} = \lambda\phi(\phi^2 - a^2) + \mu\phi\chi^2, \quad (54)$$

and

$$\frac{d\chi}{dx} = -\mu\chi\left(\frac{\chi^2}{4} - a^2\right) + \mu\phi^2\chi, \quad (55)$$

and the domain ribbon appears after setting  $\chi \rightarrow 0$ . Let us examine the behavior of the  $\chi$  field in the background of this defect. We take the parameters  $\lambda$  and  $\mu$  real and positive with  $\lambda = 4\mu$ , for simplicity. In this case, the  $\chi$  field can generate a kink inside the BPS domain ribbon. In fact,  $\phi^2 = a^2/2$  in the core of the domain ribbon, and the corresponding effective potential results to be

$$U_3(a^2/2, \chi) = \mu^2 a^6 + \frac{1}{32} \mu^2 \chi^2 (\chi^2 - 2a^2)^2. \quad (56)$$

Here a mass for the  $\chi$  field can be introduced; it is given by

$$m^2(a^2/2, \chi) = \frac{\mu^2 a^4}{4} \quad (57)$$

Outside the domain wall, for  $\phi \rightarrow 0$  the  $\chi$  boson mass is given by  $m_\chi^2(out) = \mu^2 a^4$ , and in the other outside region, where  $\phi^2 = a^2$ , the  $\chi$  boson acquire a mass  $m_\chi^2(out) = 4\mu^2 a^4$ . Then, for  $\lambda = 4\mu$  we see that the  $\phi^6$  ribbon traps the  $\chi$  field and topological defects associated with the field  $\chi$  will form inside this domain ribbon. The situation here is different from the two former cases, but the fact that  $m_\chi^2(out) > m_\chi^2(in)$  ensures the presence of the  $\chi$  particles inside the BPS ribbon.

The effective potential (56) is minimized by a field configuration for which  $\chi = 0$  or  $\chi = \pm\sqrt{2}a$ , and now inside the domain ribbon generated by the  $\phi$

field it is possible that the  $\chi$  field presents domains with  $\chi = 0$  and  $\chi^2 = 2a^2$ . We recall that this case is different from the former case, where the nested field was governed by some  $\chi^4$  potential. Here the  $\chi$  field may present symmetric and asymmetric domains, which should be connected by topological defects. We expect domains with  $\chi = \sqrt{2}a$ ,  $\chi = 0$  and  $\chi = -\sqrt{2}a$  to form at different positions, uncorrelated beyond a given coherence length  $\xi$ . Correlated domains should form, however, in conformity with the kink structure of the  $\chi^6$  system, in which energy considerations favor the presence of kinks that connect asymmetric vacua to the symmetric vacuum. To write the explicit solutions we see from  $U_3(\phi, 0)$  that the host domain ribbon appears from  $\chi = 0$  and

$$\phi^2(x) = \frac{1}{2}a^2[1 - \tanh(\lambda a^2 x)] , \quad (58)$$

and here we should set  $\lambda \rightarrow 4\mu$ , to get to the case we are considering above. For the kink, we consider  $U_3(a^2/2, \chi)$  and work on the transverse direction to obtain

$$\chi^2(y) = a^2 \left[ 1 - \tanh \left( \frac{1}{2} \mu a^2 y \right) \right] . \quad (59)$$

These are the configurations for the host domain ribbon and the internal kink in the present case, respectively. The initial separation distance between defects should be of the order  $\xi$ , the coherence length that is now given by  $\xi \approx 1/|\mu|a^2$ . Like in the former case, here we can also have a region in parameter space where stable kinks appear inside domain ribbons, and this is another example where a BPS domain ribbon hosts topological kinks.

## 4.4 Fermionic Behavior

Let us investigate the presence of fermions in the background of the BPS domain ribbon built from the  $\phi$  field in the systems introduced in the former subsections. To this end, we have to read the effective mass of the fermions in the domain ribbon background from the Yukawa couplings.

For the model of Sec. 4.1, the effective mass of the fermions in the BPS domain ribbon background (24) can be read off from the Yukawa couplings

$$\mathcal{L}_Y^1 = -\frac{3}{2}\lambda \left( \phi^2(x) - \frac{a^2}{3} \right) \bar{\psi}\psi - \frac{1}{2}\mu\phi^2(x)\bar{\rho}\rho . \quad (60)$$

From this expression we see that it is energetically favorable for the fermions  $\psi$  to reside inside the domain ribbon, whereas this is not the case for the other Majorana field. This result may perhaps lead to a mechanism for finding charged domain ribbons, provided it resists the complexification of the spinor  $\psi$ .

For the model of Sec. 4.2, the corresponding Yukawa couplings in the Lagrangian give

$$\mathcal{L}_Y^2 = -\frac{3}{2}\lambda \left( \phi^2(x) - \frac{a^2}{3} \right) \bar{\psi}\psi - \frac{1}{2}\mu \left( \phi^2(x) - \frac{a^2}{2} \right) \bar{\rho}\rho . \quad (61)$$

Thus, both fermions are trapped inside the BPS domain ribbon solution corresponding to this system. Moreover, we would like to point out that one of the Majorana fermions  $\rho$  is massless in the core of the defect and sees an isotropic exterior region. The possibility of trapping massless fermions inside the domain ribbon may open an interesting scenario: To find stable domain rings, the  $2+1$  dimensional analogue of domain bubbles in  $3+1$  dimensions, with the tension of the domain defect being equilibrated by the quantum mechanical degeneracy pressure exerted by the fermions that inhabit the ribbon.

The same analysis can be carried out in the system presented in Sec. 4.3. In that case, the Yukawa couplings are given by

$$\mathcal{L}_Y^3 = -\frac{3}{2}\lambda \left( \phi^2(x) - \frac{a^2}{3} \right) \bar{\psi}\psi - \frac{1}{2}\mu \left( \phi^2(x) + a^2 \right) \bar{\rho}\rho . \quad (62)$$

From this expression we see that it is energetically favorable for the fermions  $\psi$  to reside inside the domain ribbon, whereas this is not the case for the other Majorana field. Indeed, the fermion  $\rho$  feels an effective potential that favor one of the exterior regions of the domain ribbon. This result may perhaps lead to the formation of Fermi disks, the  $2+1$  dimensional analogue of Fermi balls in  $3+1$  dimensions, which represent a bag of false vacuum populated by a Fermi gas that stabilizes the soliton against collapse.

The above reasoning are mostly speculative and a more careful investigation should be carried out, but this is out of the scope of the present work.

However, we recall that we have just studied the Yukawa couplings in the background of the domain ribbon, for  $\chi = 0$ . This investigation is almost the same investigation one has to deal with in systems with a single field –see, for instance, Ref. [21]– for explicit calculations concerning Fermi balls in  $3 + 1$  dimensions.

## 5 Comments and Conclusions

In this work we have considered the existence of BPS topological defects with internal structure in planar systems of two coupled real scalar superfields. To this end, we have endowed the kind of systems of our interest with an extended supersymmetry and looked into the sector of extended solutions belonging to BPS representations. These configurations are nothing but supersymmetric domain ribbons that solve the set of Bogomol’nyi equations of the model. The system is entirely defined in terms of just one function of the fields, the superpotential  $\mathcal{W}$ . We have studied three different choices for  $\mathcal{W}$ , some of them leading to the appearance of several seemingly interesting kinds of topological defects: BPS domain ribbons endowed with an internal structure given by a kink.

The first system is presented to illustrate the simplest case where excitations of the bosonic field in the background of a  $\phi^6$  ribbon does not favor the formation of internal structure, although it is favorable for the fermion  $\psi$  to reside inside this domain ribbon. This result may perhaps lead to a mechanism for finding charged domain ribbons, provided it resists the complexification of the spinor  $\psi$ . The presence of the domain ribbon is shown to break one half of the supersymmetries, that is, it is a BPS state. The superpartners of this configuration can be built just by acting on it with the broken supercharges to obtain the fermionic zero modes that shall be subsequently quantized.

The superpotential that leads to the second system is suggested by symmetry considerations on the effective potential produced by the background BPS ribbon. We have shown that the modified superpotential allows the entrapment of bosons as well as the formation of  $\chi$  kinks within the core of the domain ribbon. Here, however, the defects are of different nature: The

domain ribbon comes from the  $\phi$  field (with a sixth-order potential) and connects the symmetric minimum ( $\phi = 0$ ) to an asymmetric one ( $\phi^2 = a^2$ ), while the kink is generated by the  $\chi$  field (with a fourth-order effective potential) and connects the two asymmetric minima  $\chi^2 = (\lambda/2\mu) a^2$ . It is interesting to recall that despite the asymmetric behavior of the field  $\phi$  of the ribbon at both outside regions, this asymmetry is not seen by the  $\chi$  field in this system. This new scenario allows building topological defects inside a topological defect. For a certain parameter range it becomes energetically favorable for  $\chi$  condensates to form within the core of the domain ribbon. Different domains must be separated by a region where  $\chi = 0$  that is naturally understood as the location of the resulting internal  $\chi$  kink.

The third system is defined by another potential, in which both the  $\phi$  and  $\chi$  fields present quite a similar behavior. In this case we have shown that the system admits defects of the same nature since now we can find  $\chi^6$  kinks inside a BPS  $\phi^6$  domain ribbon, in spite of the explicit asymmetry of the effective potential outside the domain ribbon.

The host domain ribbon solutions preserve one half of the supersymmetries, thus being BPS states. They are known to be classically stable [7, 8]. When we consider the inclusion of kinks inside these supersymmetric domain ribbons, unfortunately, we cannot perform a complete analysis of their stability. Nevertheless, by means of simple energetic arguments, we are able to show that there is a parameter range (in the second and third models) in which these solutions might be classically or linearly stable.

We have shown that the domain ribbon solution of the second system traps both fermions and one of them become massless in the core of the defect and sees an isotropic exterior region. This feature could lead to the formation of stable domain rings of finite radius. On the other hand, the third system has an effective potential for one of the Majorana fields that favor one of the exterior regions of the domain ribbon. This result may perhaps lead to the formation of Fermi disks in  $2 + 1$  dimensions. These possibilities, as well as the finite temperature effects on the class of systems of our interest, deserve further investigations. We hope to report on these issues elsewhere.

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